

# Synchronization on Effective Networks

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The study of network synchronization has attracted increasing attention recently. In this paper, we strictly define a class of networks, namely effective networks, which are synchronizable and orientable networks. We can prove that all the effective networks with the same size have the same spectra, and are of the best synchronizability according to the master stability analysis. However, it is found that the synchronization time for different effective networks can be quite different. Further analysis show that the key ingredient affecting the synchronization time is the maximal depth of an effective network: the larger depth results in a longer synchronization time. The secondary factor is the number of links. The more links connecting the nodes in the same layer (horizontal links) will lead to longer synchronization time, while the increasing number of links connecting nodes in neighboring layers (vertical links) will accelerate the synchronization. Our findings provide insights into the roles of horizontal and vertical links in synchronizing process, and suggest that the spectral analysis is helpful yet insufficient for the understanding of network synchronization.

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Synchronization is observed in many natural, social, physical and biological systems, and has found applications in a variety of fields [1]. As a result, the subject of synchronization is continuously calling for serious and systematic investigation, and has evolved to be an independent field of scientific research. Recently, the dramatically increasing interests in complex networks have been pervading the study of synchronization—understanding the synchronizing process of a network of dynamical systems is attracting more and more attention (see the review articles [2, 3] and the references therein). A particularly interesting issue is how to enhance the network synchronizability [4]. Under the framework of master stability analysis [5, 6], if the synchronization region is bounded and the nodes are coupled linearly and symmetrically, the network synchronizability can be measured by the eigenratio, which is defined as the ratio of the largest eigenvalue to the smallest nonzero eigenvalue of the coupling matrix. A smaller eigenratio indicates a better synchronizability, and vice versa. Some effective synchronizability-enhancement methods, aiming at reducing the eigenratio, are proposed. These methods include the discrete optimization of network structure [7, 8], the regulation of coupling pattern [9, 10, 11, 12, 13, 14], the adaptive coupling strategy [15, 16] and the structural modification [17, 18, 19]. It is worth noting, although the coupling matrix of coupling pattern regulated networks and adaptive networks may become asymmetric, which correspond to directed networks, the master stability analysis is still valid [5].

Recently, Nishikawa and Motter [20, 21] proposed a class of maximally synchronizable networks that have the smallest eigenratio,  $R = 1$ . They showed that a network is maximally synchronizable if (i) it embeds an oriented spanning tree, (ii) it does not have any directed loops, and (iii) it has normalized input coupling strength of each node. Their work [20, 21] provides a start point to the understanding of the role of directivity and feedback loops in network-based synchronization [22], and is also relevant to the synchronization control [23, 24]. However, some crucial issues, such as the time required for full synchronization, lack serious consideration thus far.

In this paper, using the language of graph theory [25], we define a new class of networks, namely *effective networks*, as the *synchronizable* and *orientable* networks, and prove that the coupling matrix of any effective network has eigenvalues  $0 = \lambda_1 < \lambda_2 = \dots \lambda_N = 1$ . We further propose a preferential selection algorithm (PSA) to extract effective network from any given connected undirected network, and investigate the synchronization of chaotic Logistic systems [26] on effective networks, emphasizing on the relation between the topology and synchronization time. Surprisingly, although the spectra of all effective networks are exactly the same, there is a great difference between their synchronization time. Based on extensive simulations, it is found that the synchronization time is mainly determined by the maximal depth of an effective network, complementary to a recent study about the synchronization of the Kuramoto model in an ideal tree [27]. The secondary factor affecting the synchronization time is the number of links. The more the links connecting the nodes in the same layer (*horizontal links*), the longer the synchronization time. Our

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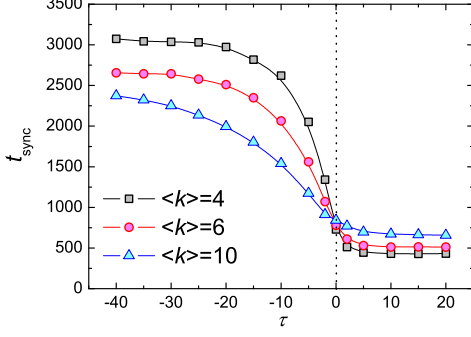


FIG. 1: (Color online)  $t_{sync}$  vs.  $\tau$  for different average degrees,  $\langle k \rangle = 4, 6$ , and  $10$ . The dot line indicates the crossover. The network size is  $1000$ , and the coupling strength takes  $0.55$ . Each data point is obtained by averaging over  $100$  independent realizations.

simulation suggests a logarithmic relation between the synchronization time and the number of those links. In contrast, the increasing number of links connecting nodes in neighboring layers (*vertical links*) will accelerate the synchronization speed. Given the maximal depth, the simulation results indicate that when the network consists of abundant horizontal and vertical links, the ratio of the number of horizontal links to the number of vertical links mainly determines the synchronization time.

We denote  $D(V, E)$  a directed graph, where  $V$  is the set of node and  $E$  the set of directed links. The multiple links and self-connections are not allowed. A directed network is called *synchronizable* if for any two different nodes  $v_i$  and  $v_j$ ,  $\exists P(v_i \rightarrow v_j)$ , or  $\exists P(v_j \rightarrow v_i)$ , or  $\exists k \neq i, j$  such that  $\exists P(v_k \rightarrow v_i)$  and  $\exists P(v_k \rightarrow v_j)$ . Here,  $P(v_i \rightarrow v_j)$  denotes an arbitrary directed path from  $v_i$  to  $v_j$ . In language, any pair of nodes are either coupled directly or driven by some common nodes. A directed network is *orientable* if it contains no directed cycles. Note that, a synchronizable network must be connected, while an orientable network is not necessarily connected. We normalize the input coupling strength of each node:  $l_{ii} = 1$ ,  $l_{ij} = -1/k_i^-$  if the directed link from  $v_j$  to  $v_i$  exists, otherwise  $l_{ij} = 0$ . Here,  $k_i^-$  denotes the in-degree of node  $v_i$ . A node is called *source* if its in-degree equals zero, accordingly, we have a *Lemma*: Any orientable network has at least one source. Proof.—Assume there exists an orientable network  $D$  containing no source. Denote  $v_1 \rightarrow v_2 \rightarrow \dots \rightarrow v_l$  the longest directed path in  $D$ . Since  $k_{v_1}^- > 0$ , there exists at least a link  $v \rightarrow v_1$ . For  $D$  contains no directed loops,  $v \neq v_2, \dots, v_l$ , thus the directed path  $v \rightarrow v_1 \rightarrow \dots \rightarrow v_l$  is even longer, which conflicts to the above assumption. Therefore,  $D$  contains at least one source. For effective network (i.e., synchronizable and orientable network), we have a *Theorem*: The normalized Laplacian of an effective network,  $L$ , has eigenvalues  $0 = \lambda_1 < \lambda_2 = \dots = \lambda_N = 1$ . Proof.—

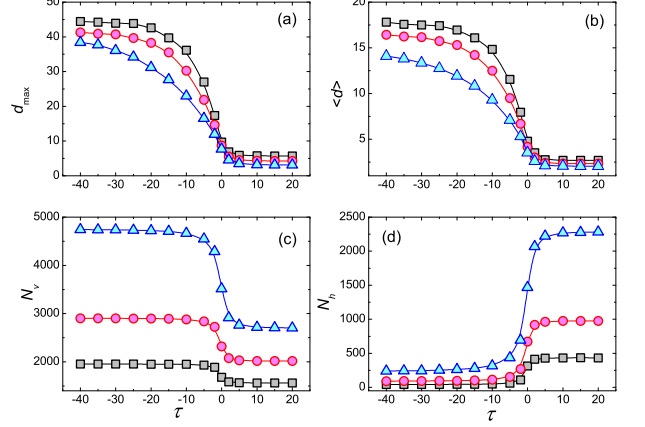


FIG. 2: (Color online) The change of (a) the maximal depth, (b) the average depth, (c) the number of vertical links,  $N_v$ , and (d) the number of horizontal links,  $N_h$ , with the parameter  $\tau$  for average degree  $\langle k \rangle = 4, 6$ , and  $10$ . The network is a BA scale-free network with  $N=1000$ . Each data point is obtained by averaging over  $100$  independent realizations. Symbols are of the same meanings as those in Fig. 1.

There exists an ordered list of  $V$  such that every link in  $E$  is from a former node to a later node. One can get this list from an empty list according to the following steps: (i) Add all sources of the current network at the top of the list in an arbitrary order; (ii) Remove those source from the current network and go to step (i) until the network becomes empty. This process is practicable since the network is finite, and after each removal of sources, the remain network is still orientable, thus has at least one new source (see the *Lemma*). Denote such an ordered list as  $Q = \{v_1, v_2, \dots, v_N\}$ , clearly, if  $i > j$ , the directed link  $(v_i \rightarrow v_j)$  can not exist because in the induced subgraph of nodes  $\{v_j, v_{j+1}, \dots, v_N\}$ ,  $v_j$  is a source without any in-link. Shift columns and rows of  $L$  corresponding to the ordered list  $Q$  and denote it by  $L_Q$ , then,  $L_Q$  is an down-triangle matrix having the same spectrum as  $L$ . Therefore, all the eigenvalues of  $L_Q$ , determined by the equation  $\det(\lambda I - L_Q) = 0$ , are its diagonal elements. Any node having in-degree larger than 0 corresponds to a unit diagonal element, while a source corresponds to a zero diagonal element. Clearly,  $v_1$  is a source, and if there exists another source  $v_k$ , since there are no directed paths end at  $v_1$  or  $v_k$ ,  $D$  is not synchronizable, conflicting to the condition that  $D$  is an effective network. Actually, any effective network has one and only one source, so the normalized coupling matrix has one zero-eigenvalue and  $N - 1$  eigenvalues all equal 1.

We propose an algorithm, namely *preferential selection algorithm* (PSA), to extract effective networks from any given connected undirected network  $G(V, E)$ . The procedure is as follows: (i) Set an empty list  $Q$ ; (ii) Select a node  $v_i$  in  $G$  with probability proportional to  $k_i^\tau$  where

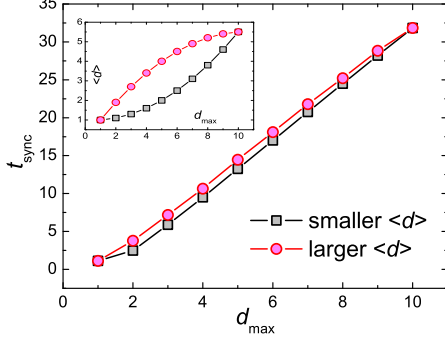


FIG. 3: (Color online) The synchronization time vs. the maximal depth. The black squares and red circles represent two different configurations of the toy model and the former has smaller average depth. The inset compares the average depth of the two configurations. The network size is  $N = 1001$  and the number of vertical links is  $N_v = 1000$ . Each data point is obtained by averaging over 1000 independent realizations.

$k_i$  is the degree of  $v_i$  and  $\tau$  is a free parameter. Put this node at the top of  $Q$ ; (iii) Pick a node  $v_j$  with probability proportional to  $k_j^\tau$ , satisfying that it does not belong to  $Q$  and at least one of its neighbors belongs to  $Q$  already. Push it at the end of  $Q$ ; (iv) Repeat the step (iii) until all nodes in  $V$  belong to  $Q$ . Since  $G$  is connected, the termination condition can always be achieved. Denoting  $Q = \{v_1, v_2, \dots, v_N\}$  where  $N$  is the size of  $G$ , then for each link  $(v_i, v_j) \in E$ , we set its direction as  $v_i \rightarrow v_j$  if  $i < j$ , or as  $v_j \rightarrow v_i$  if  $i > j$ . All those directed links constitute a set  $E'$ , and  $D(V, E')$  is an effective network. The proof is straightforward and thus is omitted here. Effective network has a clear hierarchical structure that the nodes closer to the source have higher levels. The PSA with positive  $\tau$  tends to put high-degree nodes at higher levels. For a given undirected network  $G$ , the extracted effective networks for different  $\tau$  have far different topologies, and we will see below that they exhibit far different synchronization behaviors.

We choose a benchmark individual dynamical system for the numerical analysis, namely the *Logistic map* [26]. For an arbitrary node  $v_i$ , we denote its state as  $x_i$ , which evolves as  $x_{i,t+1} = f(x_{i,t}) - \varepsilon \sum_{j=1}^N l_{ij} f(x_{j,t})$ , where  $\varepsilon$  is the overall coupling strength,  $l_{ij}$  is the element of the normalized Laplacian  $L$ ,  $f(x) = ax(1-x)$  is a Logistic map with  $a = 4.0$  fixed in this paper. According to Ref.[28], it can be proved that the effective network will achieve the complete synchronization state when the coupling strength satisfies the inequality  $1 - e^{-\lambda} < \varepsilon < 1$ , where  $\lambda$  is the Lyapunov exponent of  $f(x)$  at the current parameter.

To reveal how the topology affects the efficiency of synchronization on effective networks, we study the synchronization time versus the parameter  $\tau$ . The underlying networks before PSA are generated by Barabási-Albert

(BA) model [29]. Given a  $\tau$ , we first extract the effective network using the PSA, and then randomly assign each node an initial state between 0 and 1, and then all the nodes iterate according to the coupling equations for 2000 time steps, which is long enough to drive the system to the synchronization state. Then, we give each node a perturbation, that is, adding a small number randomly chosen in  $(-0.005, 0.005)$ . The synchronization time,  $t_{\text{sync}}$ , is defined as the time steps required for the system to return the synchronization state. In the numerical implementation, we monitor the standard deviation,  $\sigma$ , of the states of all nodes, and the system is considered to be synchronized if  $\sigma < 10^{-6}$  and will not increase any longer.

Figure 1 reports the synchronization time as a function of  $\tau$ . Clearly, the synchronization time decreases remarkably as the increasing of  $\tau$ . Although the effective networks corresponding to different  $\tau$  have exactly the same spectrum, their synchronization behaviors are greatly different and the one with higher-degree nodes in the top-per positions synchronizes faster (i.e., a larger  $\tau$  leads to a faster synchronizing process). In addition, there is a crossover of the curves for different average degrees at  $\tau_c \approx 0$  (represented by a dot line in Fig. 1). Interestingly, when  $\tau$  is larger than  $\tau_c$ , the network with smaller average degree will synchronize faster, in contrast, the network with larger average degree will synchronize faster. To understand these phenomena observed in Fig. 1, we next study the structure properties of effective networks at different  $\tau$ .

In an effective network, the number of links in the shortest path from the source node to a node  $v_i$  is called the *depth* of  $v_i$ , denoted by  $d_i$ , and the depth of the source node is zero. The *maximal depth* and the *average depth* are defined as  $d_{\text{max}} = \max\{d_i, 1 \leq i \leq N\}$  and  $\langle d \rangle = \frac{1}{N} \sum_{i=1}^N d_i$ , respectively. We classify the links into two categories: vertical links are the ones connecting two nodes with different depths (obviously, the difference is one) while horizontal links connecting two nodes with the same depth. Figure 2 reports the change of these structural measurements with  $\tau$ . With the increasing of  $\tau$ , the number of horizontal links will increase too, while the maximal depth, the average depth and the number of vertical links will decrease. Especially, the shapes of  $d_{\text{max}}$  vs.  $\tau$  and  $\langle d \rangle$  vs.  $\tau$  curves are very similar to that of  $t_{\text{sync}}$  vs.  $\tau$  curve, therefore one may guess that either  $d_{\text{max}}$  or  $\langle d \rangle$  (or both) mainly determines the synchronization time. To judge whether it is true, we propose a toy model, where  $N + 1$  nodes are connected by  $N$  vertical links starting from a single source node, resulting in a tree structure. Given the maximal depth  $d_{\text{max}}$  (in the present simulations, we tune  $d_{\text{max}}$  in the range  $1 \leq d_{\text{max}} \leq 10$ ), we (i) put  $\frac{N}{10}$  nodes in each of the  $d_{\text{max}} - 1$  layers from  $1 \leq d \leq d_{\text{max}} - 1$  and the remaining  $N(1 - \frac{d_{\text{max}} - 1}{10})$  nodes in the  $d_{\text{max}}$ -th layer; or (ii) put  $\frac{N}{10}$  nodes in each of the  $d_{\text{max}} - 1$  layers from  $2 \leq d \leq d_{\text{max}}$  and the remaining  $N(1 - \frac{d_{\text{max}} - 1}{10})$  nodes in the first layer. Each  $d$ -th layer node with  $1 \leq d \leq d_{\text{max}}$  connects to (with an in-link) the

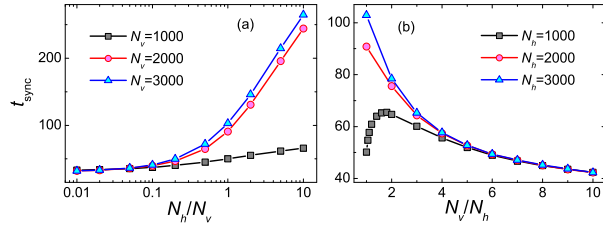


FIG. 4: (Color online) Effects of (a) the number of horizontal links and (b) the number of vertical links on the synchronization time. The network size is  $N = 1001$  and each data point is obtained by averaging over 1000 independent realizations.

node in the  $(d-1)$ -th layer one by one. Clearly, if  $d_{\text{max}} = 1$  or  $d_{\text{max}} = 10$ , the two methods lead to statistically the same  $\langle d \rangle$ , while for  $1 < d_{\text{max}} < 10$ , the former method (putting more nodes in the last layer) gives longer  $\langle d \rangle$ . In the inset of Fig. 3, we report the difference of average depths in the case of  $N = 1000$ . It is clearly observed in Fig. 3 that the synchronization time is approximately proportional to the maximal depth, while almost not affected by the average depth.

Furthermore, to reveal the roles of the vertical and horizontal links, we randomly add some vertical and horizontal links in the above-mentioned tree topologies. As shown in Fig. 4(a), the horizontal links hinder the synchronization. In the case of minimal number of vertical links (i.e.,  $N_v = 1000$ ), the horizontal links, together with vertical links, make up some triangles in the form  $\{v_i \rightarrow v_j, v_i \rightarrow v_k, v_j \rightarrow v_k\}$  with  $v_j$  and  $v_k$  in the same layer; and in the cases of abundant vertical links (i.e.,  $N_v = 2000$  and  $N_v = 3000$ ), they make up two types of triangles, respectively with  $v_j$  and  $v_k$  in the same layer and with  $v_i$  and  $v_j$  in the same layer. These triangles introduce additional disturbances among individuals, and thus slow down the synchronization. The simulations suggest a logarithmic relation between  $t_{\text{sync}}$  and  $N_h$  at large  $N_h$ .

The effects of vertical links are twofold. Firstly, more vertical links are helpful for the propagation of controlling signals from the upper-layer nodes to the lower-layer nodes. At the same time, the additional vertical links will increase the number of triangles, and thus hinder the synchronization. These two competing effects bring nontriv-

ial role of vertical links. In simulations, we find that when there are not so many horizontal links, adding vertical links will first hinder synchronization, and then enhance synchronization, leading to a maximal synchronization time (Fig. 4(b)). When the network possesses abundant horizontal links, the more vertical links will accelerate the synchronization. Simulation results in Fig. 4(a) and Fig. 4(b) suggest that when the network consists abundant vertical and horizontal links, the ratio  $N_h/N_v$  mainly determines the synchronization time: the larger the ratio, the slower the synchronizing process. This gives a qualitative explanation of the crossover observed in Fig. 1. When  $\tau > 0$ , the maximal depths of networks with different average degrees are more or less the same, but  $N_h/N_v$  in the case of  $\langle k \rangle = 10$  is much larger than the cases of  $\langle k \rangle = 4$  and  $\langle k \rangle = 6$ . Therefore, when  $\tau > 0$ , a network with larger average degree synchronizes even slower than the one with smaller average degree.

In conclusion, we have proposed a preferential selection algorithm to extract effective networks from any given connected undirected network. Although the spectra of all effective networks are the same, their synchronization time can be far different. The key ingredient affecting the synchronization time is the maximal depth of an effective network: the larger the  $d_{\text{max}}$ , the longer the  $t_{\text{sync}}$ . The secondary ingredient is the number of links. In the cases of abundant links, the smaller number of horizontal links and larger number of vertical links will lead to a faster synchronization. Actually, with fixed  $d_{\text{max}}$ , our simulation indicates that the ratio  $N_h/N_v$  mainly determines the synchronization time. Under the framework of master stability analysis [5, 6], the majority of previous studies on network synchronization only focus on the spectral analysis of the coupling matrix [2, 3], and it has been shown that synchronization time is proportional to  $1/\lambda_2$  for typical networks [30], but this is not the case for the effective networks we propose, which have exactly the same spectrum yet exhibit noticeably different synchronization behaviors. Based on the results reported here, we claim that the spectral analysis is helpful but not enough for the understanding of network synchronization. Our findings not only emphasize again the fundamental importance of directivity [20, 21], but also provide insights into the roles of horizontal and vertical links in the synchronizing process.

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